

# ON DERIVATION OF EULER-LAGRANGE EQUATIONS FOR INCOMPRESSIBLE ENERGY-MINIMIZERS

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**ABSTRACT.** We prove that any distribution  $q$  satisfying the equation  $\nabla q = \operatorname{div} \mathbf{f}$  for some tensor  $\mathbf{f} = (f_j^i)$ ,  $f_j^i \in h^r(U)$  ( $1 \leq r < \infty$ ) -the *local Hardy space*,  $q$  is in  $h^r$ , and is locally represented by the sum of singular integrals of  $f_j^i$  with Calderón-Zygmund kernel. As a consequence, we prove the existence and the local representation of the hydrostatic pressure  $p$  (modulo constant) associated with incompressible elastic energy-minimizing deformation  $\mathbf{u}$  satisfying  $|\nabla \mathbf{u}|^2, |\operatorname{cof} \nabla \mathbf{u}|^2 \in h^1$ . We also derive the system of Euler-Lagrange equations for incompressible local minimizers  $\mathbf{u}$  that are in the space  $K_{\operatorname{loc}}^{1,3}$  (defined in (1.2)); partially resolving a long standing problem. For Hölder continuous pressure  $p$ , we obtain partial regularity of area-preserving minimizers.

## 1. INTRODUCTION

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  be a bounded Lipschitz material body. For Mooney-Rivlin or Neo-Hookean materials [Ba 77], [TO 81], [Og 84], such as vulcanized rubber, in the equilibrium state, one is interested in minimizing the elastic energy

$$(1.1) \quad E[\mathbf{w}] := \int_{\Omega} L(\nabla \mathbf{w}(x)) dx,$$

for incompressible  $W^{1,2}$ -deformations  $\mathbf{w} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ , subject to its own boundary condition, and corresponding to a given smooth bulk energy  $L : \mathbb{M}^{n \times n} \rightarrow \mathbb{R}$ . Let us define the subspace  $K^{1,r}$  for  $1 \leq r < \infty$ , by

$$(1.2) \quad K^{1,r}(\Omega, \mathbb{R}^n) := \{ \mathbf{w} \in W^{1,r}(\Omega, \mathbb{R}^n) : \operatorname{cof} \nabla \mathbf{w} \in L^r(\Omega, \mathbb{M}^{n \times n}) \},$$

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where  $W^{1,r}$  denotes the usual *Sobolev spaces* (see for example, [GT 97, Chapter 7]) and  $\text{cof } P$  is the *cofactor* matrix, whose  $ij$ -th entry is  $(-1)^{i+j}$  times the determinant of  $(n-1) \times (n-1)$  submatrix obtained by deleting the  $i$ -th row and the  $j$ -th column from the  $n \times n$  matrix  $P$ . Using the identity  $P^t \text{cof } P = Id_n \det P$ , it follows that  $\det \nabla \mathbf{w} \in L^1$  for any  $\mathbf{w} \in K^{1,2}$ . Since  $|P| = |\text{cof } P|$  for any  $P \in \mathbb{M}^{2 \times 2}$ , the function spaces  $K^{1,r}$  and  $W^{1,r}$  are equal in  $\mathbb{R}^2$ . Let us denote the admissible set of deformations

$$(1.3) \quad \mathcal{A} := \{ \mathbf{w} \in K^{1,2}(\Omega, \mathbb{R}^n) : \det \nabla \mathbf{w} = 1 \text{ a.e. in } \Omega \},$$

We call  $\mathbf{u} \in \mathcal{A}$  to be a *local minimizer* of  $E[\cdot]$  if and only if

$$(1.4) \quad E[\mathbf{u}] \leq E[\mathbf{w}] \quad \text{for all } \mathbf{w} \in \mathcal{A} \text{ and } \text{supp}(\mathbf{w} - \mathbf{u}) \subset \Omega.$$

Under the hypothesis that the energy density  $L$  is smooth, *polyconvex* (convex function of minors) [Ba 77] and satisfies the growth condition

$$(1.5) \quad C_1(|X|^2 + |\text{cof } X|^2) - C_2 \leq L(X) \leq C_3(1 + |X|^2 + |\text{cof } X|^2),$$

for all  $X \in \mathbb{M}^{n \times n}$ , for some  $C_1 > 0$ ,  $C_2 \geq 0$ ,  $C_3 > 0$ , where  $|X|^2 := \text{trace}(X^t X)$ , using direct methods in the calculus of variations together with weak continuity of the determinant, J. Ball [Ba 77] proved the existence of local minimizers  $\mathbf{u} \in \mathcal{A}$  of the energy  $E[\cdot]$ . An example of polyconvex  $L$  satisfying the growth condition (1.5) is the stored-energy for incompressible isotropic Mooney-Rivlin materials in  $\mathbb{R}^3$ , given by

$$(1.6) \quad L(X) = \frac{\mu_1}{2}(I_1(X) - 3) + \frac{\mu_2}{2}(I_2(X) - 3),$$

where  $I_1(X) := \text{trace}(C) = |X|^2$ ,  $I_2(X) := \frac{1}{2} \left[ (\text{trace}(C))^2 - \text{trace}(C^2) \right] = |\text{cof } X|^2$ , are the first two principle invariants of the right Cauchy-Green strain tensor  $C := X^t X$  and  $\mu_1, \mu_2$  are positive material constants.

Though the existence of the local minimizers of  $E[\cdot]$  in  $\mathcal{A}$  is known for over 30 years, the existence of integrable hydrostatic pressure associated with such minimizers, the derivation of system of Euler-Lagrange equations, and the partial regularity for such minimizers remains a challenging open problem. In this article we prove the following results:

(I) The  $h^r$  ( $1 \leq r < \infty$ ) -integrability and local representation of any distribution  $q$  satisfying the equation  $\nabla q = \mathbf{f}$ , where  $\mathbf{f} := (f_j^i)$ ,  $f_j^i \in h^r$ , the *local  $r$ -Hardy spaces*. (Theorem 2.2)

(II) The existence of a pressure  $p \in L_{\text{loc}}^r$  if the minimizer is  $\mathbf{u} \in K_{\text{loc}}^{1,2r}$  for some  $r > 1$ . (Theorem 3.1)

- (III) The existence of a pressure  $p \in h^1$  if the minimizer  $\mathbf{u}$  satisfies the conditions  $|\nabla \mathbf{u}|^2, |\text{cof} \nabla \mathbf{u}|^2 \in h^1$ . (Theorem 3.1)
- (IV) The validity of the Euler-Lagrange equations if the minimizer is  $\mathbf{u} \in K_{\text{loc}}^{1,3}$ . (Theorem 4.1). The pair  $(\mathbf{u}, p)$  satisfies the system
 
$$(1.7) \quad \text{div} [DL(\nabla \mathbf{u}(x)) - p(x) \text{cof}(\nabla \mathbf{u}(x))] = 0 \quad \text{in } \Omega,$$
 where the divergence is taken in each rows.
- (V) The partial regularity of  $W^{1,3}$  area-preserving minimizers  $\mathbf{u}$  for which the hydrostatic pressure  $p$  is Hölder continuous with exponent  $0 < \alpha < 1$ . (Theorem 5.1)

The  $L^2$ -version of the result in (I) is classical (see, [Te 01, Remark 1.4, p 11]), and plays an important role in incompressible fluids [Te 01]. The result in (I) is a crucial ingredient in proving (II) & (III). The  $h^1$ -version of (I) is quite delicate and to the best of our knowledge, it is new and may be of independent interest. For the case  $r > 1$ , it follows that  $\nabla q \in W^{-1,r}$ , and adapting the classical functional-analytic approach demonstrated for  $r = 2$  (see [Te 01], [TO 81]), or arguing directly by duality, and solving the equation of the type

$$\text{div } \mathbf{w} = f \quad \text{in } V \subset \subset U, \quad \mathbf{w} = 0 \quad \text{in } \partial V,$$

[Ev 98, p. 472-474], one can prove that  $q \in L_{\text{loc}}^r$ . However, both of these approaches fail to give informations for the critical case  $r = 1$  and does not give a representation of  $q$ . Whereas, our unified singular integral approach is self-contained, simple and provides the local  $h^r$ -estimate, as well as the local representation of  $q$ . The main ideas in our proof is to represent the localized-mollified distribution  $q$  in terms of the Newtonian potential in  $\mathbb{R}^n$  and finding its uniform bound in  $h^r$ , by using Calderón-Zygmund estimate [FS 72], [CZ 52]. Finally we show that the local representation of  $q$  consists the sum of Calderón-Zygmund type singular integrals of the tensor  $\mathbf{f}$  (see equation (2.27) in Section 4).

For the case  $n = 2$ , under the stronger hypothesis that the local minimizers of  $E[\cdot]$  are classical ( $C^{1,\alpha}$ -diffeomorphism), namely in the Sobolev space  $W^{2,r}$  for some  $r > 2$ , LeTallec and Oden [TO 81] established the system of equations in (1.7). For  $n = 2$ , Bauman, Owen and Phillips [BOP 92] proved that if a minimizer is in  $W^{2,r}$  for some  $r > 2$ , then it is smooth. For such  $W^{2,r}$ ,  $r > 2$  minimizers, the authors in [BOP 92] argued directly on the level of the Euler-Lagrange equations exploring the existence of integrable hydrostatic pressure. Evans and Gariepy [EG 99]

proved that any *non-degenerate*, Lipschitz area-preserving local minimizers of  $E[\cdot]$  are  $C^{1,\alpha}(\Omega_0)$ , for some  $0 < \alpha < 1$  for a dense open subset  $\Omega_0 \subset \Omega$ . We believe that the Euler-Lagrange equations (1.7) that we derived for  $K^{1,3}$ -minimizers may be useful in understanding the partial regularity of such minimizers, as evidenced by the result in (V).

In order to prove the existence of an integrable pressure  $p$  associated with the local minimizer  $\mathbf{u}$ , we only require the additional mild assumption  $|\nabla \mathbf{u}|^2 \log(2 + |\nabla \mathbf{u}|^2), |\operatorname{cof} \nabla \mathbf{u}|^2 \log(2 + |\operatorname{cof} \nabla \mathbf{u}|^2) \in L^1_{\text{loc}}$ . For  $n = 2$ , to derive the system of equilibrium equations (1.7) for  $(\mathbf{u}, p)$  in  $\Omega$ , we need  $\mathbf{u}$  to be in  $W^{1,3}$ , whereas the best-known previous result in this direction were for  $W^{2,r}$ -minimizers for some  $r > 2$ .

We organize the paper as follows. In Section 2 we prove (I); in Section 3 we prove (II) & (III); in Section 4 we prove (IV), and finally in Section 5 we prove (V). Throughout this article  $C$  is a generic absolute constant depending on  $n, U, \Omega, \mathbf{u}(\Omega), V \subset \subset \mathbf{u}(\Omega), r$ , and  $L$ . Its value can vary from line to line, but each line is valid with  $C$  being a pure positive number.

## 2. LOCAL INTEGRABILITY OF SOLUTIONS $\nabla q = \operatorname{div} \mathbf{f}$

We recall some of the basic definitions and terminologies of Hardy spaces. Let  $1 \leq r < \infty$ . A distribution  $f$  belongs to  $H^r(\mathbb{R}^n)$  if and only if  $f \in L^r(\mathbb{R}^n)$  and  $R_j(f) \in L^r(\mathbb{R}^n)$  (see for example, [St 93, Proposition 3, p. 123]) for  $j = 1, \dots, n$ , where  $R_j$  is the Riesz transform of  $f$  given by

$$R_j(f)(x) := \lim_{\varepsilon \rightarrow 0} c_n \int_{|y| \geq \varepsilon} \frac{y_j}{|y|^{n+1}} f(x - y) dy, \quad c_n := \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{(n+1)/2}},$$

so that  $\widehat{R_j(f)}(\xi) = i \frac{\xi_j}{|\xi|} \widehat{f}$ . In short, we will write  $H^r(\mathbb{R}^n)$  as simply  $H^r$ . For  $f \in H^r$ , the norm is defined as

$$\|f\|_{H^r} := \|f\|_{L^r} + \sum_{j=1}^n \|R_j(f)\|_{L^r}.$$

A standard result [St 70, p. 237] states that a positive function  $f$ , the Riesz transform  $R_j f \in L^1_{\text{loc}}$  if and only if  $f \log(2 + f) \in L^1_{\text{loc}}$ . For  $1 < r < \infty$ , a classical result asserts that  $f \in H^r$  if and only if  $f \in L^r$ , see [St 70, p. 220]. The celebrated Fefferman duality theorem [Fe 71], [FS 72, Theorem 2], [St 93, Theorem 1, p. 142] asserts that the dual of  $H^1$  is the BMO, the functions of bounded mean oscillations. The following

theorem is due to Calderón-Zygmund [CZ 52], Stein [St 70, Theorem 3, p. 39], and Stein-Fefferman [FS 72, Corollary 1, p. 149-151].

**Theorem 2.1 (Calderón-Zygmund, Fefferman-Stein).** *Let  $1 \leq r < \infty$  and  $f \in H^r$ . Let  $G$  be a  $C^1$  function on  $\mathbb{R}^n \setminus \{0\}$  homogeneous of degree 0 with mean value 0 over the unit sphere  $\mathbb{S}^{n-1}$ , that is*

$$(2.1) \quad \int_{\mathbb{S}^{n-1}} G(x) d\sigma(x) = 0.$$

*Then the function defined as*

$$(2.2) \quad T_0 f(x) := \lim_{\delta \rightarrow 0} \int_{|y| \geq \delta} \frac{G(y)}{|y|^n} f(x - y) dy$$

*exists a.e. and furthermore,*

$$(2.3) \quad \|T_0 f\|_{H^r} \leq C_{n,r} \|f\|_{H^r}.$$

In particular,  $R_j$ 's are bounded linear operator on  $H^r$ , for any  $1 \leq r < \infty$ . Let us recall the definition of *local Hardy spaces* introduced by Goldberg [Go 79]. A distribution  $f$  on  $\mathbb{R}^n$  is said to be in the local  $r$ -Hardy space, written as  $f \in h^r$ , if and only if the maximal function

$$\mathcal{M}_{\text{loc}} f(x) := \sup_{0 < \varepsilon < 1} |(\rho_\varepsilon * f)(x)|$$

is in  $L^r$ , where  $\rho_\varepsilon := \varepsilon^{-n} \rho(x/\varepsilon)$ , is a standard approximation of the identity. The  $h^r$  norm of  $f$  is defined to be the  $L^r$  norm of the maximal function  $\mathcal{M}_{\text{loc}} f$ . It follows that if  $f \in h^r$  then  $\eta f \in h^r$  for any smooth cut-off function and  $H^r \subset h^r$ . For bounded Lipschitz domain  $\Omega \subset \mathbb{R}^n$ , we adopt the definition of Hardy spaces  $h^r(\Omega)$  introduced by Miyachi [Mi 90]. A distribution  $f$  on  $\Omega$  is said to be in  $h^r(\Omega)$  if  $f$  is the restriction to  $\Omega$  of a distribution  $F$  in  $h^r(\mathbb{R}^n)$ , i.e.,

$$\begin{aligned} h^r(\Omega) &:= \{f \in \mathcal{D}'(\Omega) : \exists F \in h^r(\mathbb{R}^n), \text{ such that } F|_\Omega = f\} \\ &= h^r(\mathbb{R}^n) / \{F \in h^r(\mathbb{R}^n) : F = 0 \text{ on } \Omega\}. \end{aligned}$$

The norm on this space is the quotient norm: the infimum of  $h^r$  norms of all possible extensions of  $f$  in  $\mathbb{R}^n$ . For  $1 < r < \infty$  the spaces  $h^r(\Omega)$  is equivalent to  $L^r(\Omega)$ . For smooth bounded domains  $\Omega$ , the Theorem 2.1 is valid for  $f \in h^1(\Omega)$ , see [Mi 90], [CKS 93].

**Theorem 2.2.** *Let  $U \subset \mathbb{R}^n$ ,  $n \geq 2$  be a bounded Lipschitz domain and  $1 \leq r < \infty$ . Let  $\mathbf{f} = (f_j^i)$  such that  $f_j^i \in h^r(U)$ , for  $1 \leq i, j \leq n$ . Then the distribution  $q : C_0^\infty(U) \rightarrow \mathbb{R}$  defined by*

$$(2.4) \quad \nabla q = \operatorname{div} \mathbf{f} \iff \langle \nabla q, \mathbf{v} \rangle = - \int_U \mathbf{f}(x) : \nabla \mathbf{v}(x) dx$$

for  $\mathbf{v} \in C_0^\infty(U, \mathbb{R}^n)$ , is in  $h^r(V)$ , for any  $V \subset\subset U$  where  $A : B := \operatorname{trace}(A^t B) = \sum_{ij} a_j^i b_j^i$ , for  $A, B \in \mathbb{M}^{n \times n}$ . Furthermore,  $q$  is locally represented by sum of singular integrals of  $f_j^i$  (see equation (2.27)), and for any  $V \subset\subset U$ , there exists  $C > 0$ , depending only on  $U$ ,  $V$  and  $r$  such that

$$\|q\|_{h^r(V)} \leq C \|\mathbf{f}\|_{h^r(V)}.$$

**Proof of Theorem 2.2.** Let  $U \subset \mathbb{R}^n$ ,  $n \geq 2$  be a Lipschitz domain. Let  $\mathbf{f} := (f_j^i) \in \mathbb{M}^{n \times n}$  and  $f_j^i \in h^r(U)$ , for  $1 \leq r < \infty$  and  $1 \leq i, j \leq n$ . Let  $q \in \mathcal{D}'(U)$ , such that

$$(2.5) \quad \nabla q = \operatorname{div} \mathbf{f} \quad \text{in } \mathcal{D}'(U).$$

Our idea is to mollify the equations in (2.5) and obtain uniform bound for the mollified  $q$ , by using Calderón-Zygmund estimate. Let  $V \subset\subset U$  be a sub-domain and  $0 < \varepsilon < \operatorname{dist}(V, \partial U)$ . Let  $\rho_\varepsilon$  be the usual mollification kernel, and define convolution  $q_\varepsilon : V \rightarrow \mathbb{R}$  by

$$q_\varepsilon(x) = (q * \rho_\varepsilon)(x) := \langle q, (\rho_\varepsilon)_x \rangle \quad \text{for } x \in V, \quad \text{where } (\rho_\varepsilon)_x(y) := \rho_\varepsilon(y - x), \quad y \in U$$

Then by the standard properties of the mollification [DL 88, Proposition 1, p492],  $q_\varepsilon$  is smooth and for any  $1 \leq i \leq n$

$$\frac{\partial}{\partial x_i}(q * \rho_\varepsilon) = \frac{\partial q}{\partial x_i} * \rho_\varepsilon = q * \frac{\partial \rho_\varepsilon}{\partial x_i}.$$

Hence mollifying the system of equations in (2.5), we obtain

$$(2.6) \quad \nabla q_\varepsilon = \operatorname{div} \mathbf{f}_\varepsilon \quad \text{in } V,$$

where the divergence is taken in each rows of  $\mathbf{f}_\varepsilon := \left( (f_j^i)_\varepsilon \right)$ , and  $(f_j^i)_\varepsilon := f_j^i * \rho_\varepsilon$  is the mollification of  $\mathbf{f}$ . Since  $f_j^i \in h^r(U)$ , we conclude that

$$(2.7) \quad (f_j^i)_\varepsilon \rightarrow f_j^i \quad \text{strongly in } h^r(V) \quad \text{as } \varepsilon \rightarrow 0,$$

for all  $1 \leq i, j \leq n$ . Applying the divergence operator to the both sides of the above equation, we obtain

$$(2.8) \quad \Delta q_\varepsilon = \operatorname{div}(\operatorname{div} \mathbf{f}_\varepsilon) \quad \text{in } V.$$

Since there is no control on the boundary values, we need to localize the equation (2.8). Let  $W \subset\subset V \subset\subset U$ . Let  $\eta \in C_0^\infty(\mathbb{R}^n)$ ,  $0 \leq \eta \leq 1$  be a cut-off function such that  $\eta \equiv 1$  in  $W$  and  $\eta \equiv 0$  outside  $V$ . Let  $\bar{q}_\varepsilon := \eta q_\varepsilon$  be the localization of  $q_\varepsilon$ . Then  $\bar{q}_\varepsilon$  is the solution of Poisson equation

$$(2.9) \quad \Delta \bar{q}_\varepsilon = \bar{f}_\varepsilon \quad \text{in } \mathbb{R}^n,$$

where

$$(2.10) \quad \begin{aligned} \bar{f}_\varepsilon &:= \eta \Delta q_\varepsilon + 2 \langle \nabla q_\varepsilon, \nabla \eta \rangle + q_\varepsilon \Delta \eta \\ &= \eta \operatorname{div}(\operatorname{div} \mathbf{f}_\varepsilon) + 2 \langle \operatorname{div} \mathbf{f}_\varepsilon, \nabla \eta \rangle + q_\varepsilon \Delta \eta. \end{aligned}$$

Therefore  $\bar{q}_\varepsilon$  is represented by the Newtonian potential of in  $\mathbb{R}^n$ . In other words,

$$(2.11) \quad \bar{q}_\varepsilon(x) = - \int_{\mathbb{R}^n} \Phi(x-y) \bar{f}_\varepsilon(y) dy,$$

where  $\Phi$  is fundamental solution of the Laplace equation in  $\mathbb{R}^n$  and is given by

$$(2.12) \quad \Phi(x) := \begin{cases} -\frac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \end{cases}$$

for  $x \in \mathbb{R}^n \setminus \{0\}$ , and  $\alpha(n) := \frac{\pi^{n/2}}{\Gamma(\frac{n}{2}+1)}$  is the volume of the unit ball in  $\mathbb{R}^n$ . Using (2.10) in (2.11), we obtain

$$(2.13) \quad \begin{aligned} \bar{q}_\varepsilon(x) &= - \int_{\mathbb{R}^n} \eta(y) \Phi(x-y) \operatorname{div}(\operatorname{div} \mathbf{f}_\varepsilon) dy \\ &\quad + 2 \int_{\mathbb{R}^n} (\langle \operatorname{div} \mathbf{f}_\varepsilon, \nabla \eta \rangle + q_\varepsilon \Delta \eta) \Phi(x-y) dy \\ &:= -I_\varepsilon^1(x) - 2I_\varepsilon^2(x) - I_\varepsilon^3(x), \end{aligned}$$

where

$$(2.14) \quad I_\varepsilon^1(x) := \int_{\mathbb{R}^n} \eta(y) \Phi(x-y) \operatorname{div}(\operatorname{div} \mathbf{f}_\varepsilon(y)) dy$$

$$(2.15) \quad I_\varepsilon^2(x) := \int_{\mathbb{R}^n} \langle \operatorname{div} \mathbf{f}_\varepsilon(y), \nabla \eta(y) \rangle \Phi(x-y) dy$$

$$(2.16) \quad I_\varepsilon^3(x) := \int_{\mathbb{R}^n} q_\varepsilon(y) \Delta \eta(y) \Phi(x-y) dy$$

By direct computations, observe that, for  $1 \leq i, j \leq n$

$$(2.17) \quad (\eta \Phi)_{y_i} = \eta_{y_i} \Phi(y) - \frac{1}{\omega_n} \frac{\eta y_i}{|y|^n},$$

$$(2.18) \quad (\eta \Phi)_{y_i y_j} = \eta_{y_i y_j} \Phi(y) - \frac{1}{\omega_n} \frac{y_i \eta_{y_j} + y_j \eta_{y_i}}{|y|^n} - \frac{1}{\omega_n} \left( \delta_{ij} - n \frac{y_i y_j}{|y|^2} \right) \frac{\eta}{|y|^n},$$

where  $\delta_{ij}$  is the Kröneckel delta and  $\omega_n := n\alpha_n$  is the surface area of the unit sphere  $\mathbb{S}^{n-1}$ . We now establish an uniform local  $h^r$ -estimates ( $1 \leq r < \infty$ ) for  $q_\varepsilon$  through the following steps.

**Step 1: Limit of  $I_\varepsilon^3$ .** Let us fix  $x \in W \subset\subset V \subset\subset U$ . Since  $\Delta\eta = 0$  on  $W$ , the integrand in  $I_\varepsilon^3(x)$  is smooth. Since  $q_\varepsilon$  is determined up to a constant, we can add a constant to  $y \mapsto \Delta\eta(y)\Phi|x-y|$ , if nessecary, to ensure that it has vanishing integral. For each fixed  $x \in W$ , let  $\mathbf{v}_x : V \rightarrow \mathbb{R}^n$  be the solution of the Dirichlet problem

$$(2.19) \quad \begin{cases} \operatorname{div} \mathbf{v}_x(y) = \Delta\eta(y)\Phi(x-y) & \text{for } y \in V \\ \mathbf{v}_x = 0 & \text{on } \partial V. \end{cases}$$

Then using (2.19), integrating by parts, and the convergence of  $\mathbf{f}_\varepsilon$  in (2.16), we obtain

$$(2.20) \quad \begin{aligned} I_\varepsilon^3(x) &= \int_{\mathbb{R}^n} q_\varepsilon(y) \Delta\eta(y) \Phi(x-y) dy \\ &= \int_{\mathbb{R}^n} q_\varepsilon(y) \operatorname{div} \mathbf{v}_x(y) dy \\ &= - \int_{\mathbb{R}^n} \langle \nabla q_\varepsilon(y), \mathbf{v}_x(y) \rangle dx \\ &= - \int_{\mathbb{R}^n} \langle \operatorname{div} \mathbf{f}_\varepsilon(y), \mathbf{v}_x(y) \rangle dy \\ &= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \nabla_y \mathbf{v}_x(y) dy \\ &\rightarrow \int_{\mathbb{R}^n} \mathbf{f}(y) : \nabla_y \mathbf{v}_x(y) dy \quad \text{as } \varepsilon \rightarrow 0 \\ &:= I_0^3(x) \quad \text{for } x \in W \subset\subset V. \end{aligned}$$

Since  $\mathbf{f}_\varepsilon \rightarrow \mathbf{f}$  strongly in  $h^r(V, \mathbb{M}^{n \times n})$ , it follows that  $I_\varepsilon^3 \rightarrow I_0^3$  strongly in  $h^r(W)$ .



**Step 2: Limit of  $I_\varepsilon^2$ .** Let us fix  $x \in W \subset\subset V \subset\subset U$ . Integrating by parts, invoking (2.17) and letting  $\varepsilon \rightarrow 0$  we have

$$\begin{aligned}
 (2.21) \quad I_\varepsilon^2(x) &= \int_{\mathbb{R}^n} \left\langle \operatorname{div} \mathbf{f}_\varepsilon(y), \Phi(x-y) \nabla \eta(y) \right\rangle dy \\
 &= - \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon : \nabla_y \left( \Phi(x-y) \nabla \eta \right) dy \\
 &= - \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon : \left( \Phi(x-y) \nabla^2 \eta - \frac{(y-x) \otimes \nabla \eta}{\omega_n |y-x|^n} \right) dy \\
 &\rightarrow - \int_{\mathbb{R}^n} \mathbf{f} : \left( \Phi(x-y) \nabla^2 \eta - \frac{(y-x) \otimes \nabla \eta}{\omega_n |y-x|^n} \right) dy \\
 &:= I_0^2(x) \quad x \in W.
 \end{aligned}$$

Using the strong convergence of  $\mathbf{f}_\varepsilon$  in  $h^r(V)$ , again it follows that  $I_\varepsilon^2 \rightarrow I_0^2$  in  $h^r(W)$ .

**Step 3: Limit of  $I_\varepsilon^1$ .** Integrating by parts twice the integral in (2.14) and using (2.18)

$$\begin{aligned}
 I_\varepsilon^1(x) &= \int_{\mathbb{R}^n} \operatorname{div} \operatorname{div} \mathbf{f}_\varepsilon(y) \eta(y) \Phi(x-y) dy \\
 &= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \nabla_y^2 \left( \eta(y) \Phi(x-y) \right) dy \\
 &= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \left( \Phi(x-y) \nabla^2 \eta(y) - \frac{1}{\omega_n} \frac{\nabla \eta \otimes (y-x) + (y-x) \otimes \nabla \eta}{|x-y|^n} \right) dy \\
 &\quad - \frac{1}{\omega_n} \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon(y) : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{\eta}{|x-y|^n} dy \\
 &:= I_\varepsilon^{11}(x) + I_\varepsilon^{12}(x), \quad x \in W,
 \end{aligned}$$

where  $Id_n$  is the  $n \times n$  identity matrix. Using the convergence of  $\mathbf{f}_\varepsilon$ , observe that as  $\varepsilon \rightarrow 0$ ,

$$\begin{aligned}
 I_\varepsilon^{11}(x) &:= \int_{\mathbb{R}^n} \mathbf{f}_\varepsilon : \left( \Phi(x-y) \nabla^2 \eta - \frac{\nabla \eta \otimes (y-x) + (y-x) \otimes \nabla \eta}{\omega_n |x-y|^n} \right) dy \\
 &\rightarrow \int_{\mathbb{R}^n} \mathbf{f} : \left( \Phi(x-y) \nabla^2 \eta - \frac{\nabla \eta \otimes (y-x) + (y-x) \otimes \nabla \eta}{\omega_n |x-y|^n} \right) dy \\
 (2.22) \quad &:= I_0^{11}(x), \quad x \in W.
 \end{aligned}$$

In order to estimate  $I_\varepsilon^{12}$ , define the kernels  $\Omega_{ij} : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  by

$$(2.23) \quad \Omega_{ij}(y) := \delta_{ij} - n \frac{y_i y_j}{|y|^2}, \quad y \in \mathbb{R}^n \setminus \{0\}, \quad i, j = 1, \dots, n.$$

Since  $n\alpha_n = \omega_n$ , integrating by parts, observe that for any  $i, j = 1, \dots, n$ ,

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} \Omega_{ij}(y) d\sigma(y) &= \int_{\mathbb{S}^{n-1}} (\delta_{ij} - ny_i y_j) d\sigma(y) \\
&= \omega_n \delta_{ij} - n \int_{\mathbb{S}^{n-1}} y_i y_j d\sigma(y) \\
&= \omega_n \delta_{ij} - n \int_{B_1} \frac{\partial}{\partial y_j} y_i dy \\
&= \omega_n \delta_{ij} - n \delta_{ij} \alpha_n \\
&= 0.
\end{aligned}$$

Hence each  $\Omega_{ij}$  satisfies all the conditions of Calderón-Zygmund Kernel [St 70]. Therefore,

$$(2.24) \quad I_\varepsilon^{12}(x) := -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \eta \mathbf{f}_\varepsilon : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n}$$

is the sum of Calderón-Zygmund type singular integrals with the homogeneous kernel  $\Omega_{ij}$ . Since  $\mathbf{f} \in h^r(U, \mathbb{M}^{n \times n})$ ,  $1 \leq r < \infty$ , by Theorem 2.1  $I^{12} \in h^r(W)$ . Furthermore, the following sum of singular integrals

$$(2.25) \quad I_0^{12}(x) := -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \eta \mathbf{f} : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n}$$

exists for almost every  $x \in W \subset\subset V$  and is in  $h^r(W)$ . From the singular integrals (2.24) and (2.25), by Theorem 2.1, we have

$$I_\varepsilon^{12}(x) - I_0^{12}(x) = -\frac{1}{\omega_n} \sum_{i,j=1}^n \int_{\mathbb{R}^n} \left( \eta(f_j^i)_\varepsilon(y) - \eta f_j^i(y) \right) \frac{\Omega_{ij}(x-y)}{|x-y|^n} dy.$$

Hence there exists  $C := C(V, W, r) > 0$  such that

$$(2.26) \quad \|I_\varepsilon^{12} - I_0^{12}\|_{h^r(W)} \leq C \sum_{j=1}^n \|(f_j^i)_\varepsilon - f_j^i\|_{h^r(V)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

**Step 4: Explicit representation of  $q$ .** To complete the proof, let us define the potential  $q : W \rightarrow \mathbb{R}$  by

$$q(x) := -(I_0^{11}(x) + I_0^{12}(x) + 2I_0^2(x) + I_0^3(x)).$$

Then from (2.20), (2.21), (2.22), and (2.26), we conclude that  $q_\varepsilon \rightarrow q$  strongly in  $h_{\text{loc}}^r$  for any  $1 \leq r < \infty$ , and hence  $q$  is represented as

$$(2.27) \quad q(x) = \int_U \mathbf{f} : (\Phi(x-y) \nabla^2 \eta - \nabla_y \mathbf{v}_x) dy \\ + \frac{1}{\omega_n} \int_U \mathbf{f} : \left( \nabla \eta \otimes (y-x) - (y-x) \otimes \nabla \eta \right) \frac{dy}{|x-y|^n} \\ + \frac{1}{\omega_n} \int_U \eta \mathbf{f} : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n}$$

for any  $x \in W$ . Since  $q$  is the strong limit of the family  $q_\varepsilon$  in  $W$ , it is independent of the choice of the cut-off function  $\eta$ . This completes the proof of Theorem 1.1.  $\square$

### 3. FIRST VARIATION OF ENERGY AND THE EXISTENCE OF HYDROSTATIC PRESSURE

Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$  be a smooth, simply connected and bounded domain and let  $L : \mathbb{M}^{n \times n} \rightarrow \mathbb{R}$  be smooth function. We are now in a position to establish the existence of integrable hydrostatic pressure associated with the local minimizers of the energy

$$(3.1) \quad E[\mathbf{w}] := \int_\Omega L(\nabla \mathbf{w}(x)) dx,$$

for incompressible  $W^{1,2}$ -deformations  $\mathbf{w} : \Omega \subset \mathbb{R}^n \rightarrow \mathbb{R}^n$ . By direct computation, observe that Mooney-Rivlin bulk-energy given by

$$(3.2) \quad L(X) = \frac{\mu_1}{2} (|\nabla \mathbf{u}|^2 - 3) + \frac{\mu_2}{2} (|\text{cof} \nabla \mathbf{u}|^2 - 3),$$

satisfies the following.

$$DL = \mu_1 P + \mu_2 \begin{pmatrix} \text{cof}(SQ)_1^1 : (SP)_1^1 & -\text{cof}(SQ)_2^1 : (SQ)_2^1 & \text{cof}(SQ)_3^1 : (SP)_3^1 \\ -\text{cof}(SQ)_1^2 : (SP)_1^2 & \text{cof}(SQ)_2^2 : (SP)_2^2 & -\text{cof}(SQ)_3^2 : (SP)_3^2 \\ \text{cof}(SQ)_1^3 : (SP)_1^3 & -\text{cof}(SQ)_2^3 : (SP)_2^3 & \text{cof}(SQ)_3^3 : (SP)_3^3 \end{pmatrix},$$

where  $Q := \text{cof} P$ , and  $(SX)_j^i$  is the  $2 \times 2$  submatrix obtained by deleting the  $i$ -th row and the  $j$ -th column of the matrix  $X \in M^{3 \times 3}$ . Furthermore,

the Cauchy-Green strain tensor is given by

$$(DL(P))^t P = \mu_1 P^t P + \mu_2 \begin{pmatrix} |Q_2|^2 + |Q_3|^2 & -\langle Q_1, Q_2 \rangle & -\langle Q_1, Q_3 \rangle \\ -\langle Q_1, Q_2 \rangle & |Q_1|^2 + |Q_3|^2 & -\langle Q_2, Q_3 \rangle \\ -\langle Q_1, Q_2 \rangle & -\langle Q_2, Q_3 \rangle & |Q_1|^2 + |Q_2|^2 \end{pmatrix}$$

for all  $P \in \mathbb{M}^{3 \times 3}$ , where  $Q_i := (\text{cof } P)_i := ((\text{cof } P)_1^i, (\text{cof } P)_2^i, (\text{cof } P)_3^i)$  be the  $i$ -th row of  $\text{cof } P$ ,  $i = 1, 2, 3$ . Motivated by the above calculations, assume that  $L$  satisfies the following growth condition.

$$(3.3) \quad \max \left( |L(P)|, |(DL(P))^t P| \right) \leq C(1 + |P|^2 + |\text{cof } P|^2),$$

for some  $C > 0$ , for any  $P \in \mathbb{M}^{n \times n}$ .

Now we prove the existence of an integrable hydrostatic pressure  $q$  on the deformed domain  $\mathbf{u}(\Omega)$  and establish an explicit representation of the pressure  $q$  in terms of Calderón-Zygmund type singular integrals of the Cauchy-Green strain  $\tilde{\sigma} := (DL(\nabla \mathbf{u}))^t \nabla \mathbf{u} \circ \mathbf{u}^{-1}$  in  $\mathbf{u}(\Omega)$ . Our proof consists of deriving the first variation of the energy  $E[\cdot]$ , obtaining the equation  $\nabla q = \text{div } \tilde{\sigma}$ , and then finally use Theorem 2.2.

**Theorem 3.1.** *Let  $L : \mathbb{M}^{n \times n} \rightarrow \mathbb{R}$  be smooth and satisfies the growth condition (3.3). Assume that  $\mathbf{u} \in \mathcal{A}$  be a continuous and injective local minimizer of  $E[\cdot]$ , such that  $|\nabla \mathbf{u}|^2, |\text{cof } \nabla \mathbf{u}|^2 \in h_{\text{loc}}^r(\Omega)$  for some  $1 \leq r < \infty$ . Then there exists a scalar function  $q \in h_{\text{loc}}^r(\mathbf{u}(\Omega))$ , such that*

$$\|q\|_{h^r(V)} \leq C \left( \|\nabla \mathbf{u}\|^2_{h^r(\mathbf{u}^{-1}(V))} + \|\text{cof } \nabla \mathbf{u}\|^2_{h^r(\mathbf{u}^{-1}(V))} \right), \quad V \subset \subset \mathbf{u}(\Omega),$$

for some  $C > 0$  (depending on  $r$ ,  $V$ ,  $n$  and  $\mathbf{u}(\Omega)$ ) and the pair  $(\mathbf{u}, q)$  satisfies the integral identity

$$(3.4) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u}) \, dx = \int_{\mathbf{u}(\Omega)} q(y) \, \text{div } \mathbf{v}(y) \, dy$$

for all  $\mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^n)$ , where  $A : B := \text{tr}(A^t B) = \sum_{i,j=1}^n a_j^i b_j^i$  is the scalar product on  $\mathbb{M}^{n \times n}$ .

*Remark 3.2.* Let  $W \subset\subset V \subset\subset \mathbf{u}(\Omega)$ , and  $\eta \in C_0^\infty(V)$  be a cut-off function such that  $\eta \equiv 1$  on  $W$ . Then  $q$  is locally represented as

$$(3.5) \quad q(x) = \int_V \tilde{\sigma} : (\Phi(x-y) \nabla^2 \eta - \nabla_y \mathbf{v}_x) dy \\ + \frac{1}{\omega_n} \int_V \tilde{\sigma} : \left( \nabla \eta \otimes (y-x) - (y-x) \otimes \nabla \eta \right) \frac{dy}{|x-y|^n} \\ + \frac{1}{\omega_n} \int_V \eta \tilde{\sigma} : \left( Id_n - n \frac{(y-x) \otimes (y-x)}{|x-y|^2} \right) \frac{dy}{|x-y|^n},$$

for any  $x \in W$ , where  $\Phi$  is Newtonian potential in  $\mathbb{R}^n$  defined in (2.12) and  $\mathbf{v}_x$  as defined in (2.19).

*Remark 3.3.* In the study of regularity of finite energy deformations, Šverák [Sv 88] proved that for any  $W^{1,n}$ -deformation  $\mathbf{w}$  with  $\det \nabla \mathbf{w}(x) > 0$ , a.e., there exists a continuous function  $\omega$  on  $\mathbb{R}$  with  $\omega(0) = 0$  such that

$$|\mathbf{w}(x) - \mathbf{w}(y)| \leq \omega(|x-y|), \quad \text{for any } x, y \in \Omega \subset\subset \mathbb{R}^n.$$

It is also well-known any  $W^{1,n}$ -deformation  $\mathbf{w}$  for which the *distortion* function  $K(\cdot, \mathbf{w}) := |\nabla \mathbf{w}(\cdot)|^n / \det \nabla \mathbf{w}(\cdot) \in L^r$  for some  $r > n-1$ , is a homeomorphism. Thus in particular, area-preserving  $W^{1,r}$  ( $r > 2$ )-deformations in the plane are continuous and open maps. However, in general for  $n \geq 3$ , any deformation  $\mathbf{w} \in K^{1,2}$  may be totally discontinuous, see [Sv 88, p. 119].

In order to prove Theorem 3.1, we establish the following first variation of the energy integral  $E[\cdot]$ .

**Lemma 3.4. First Variation.** *Let  $\mathbf{u} \in \mathcal{A}$  be a local minimizer of  $E[\cdot]$ . We further assume that  $\mathbf{u}$  is a continuous and an injective map. Then  $\mathbf{u}$  satisfies the following integral identity*

$$(3.6) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx = 0,$$

for all smooth, compactly supported and divergence free vector fields  $\mathbf{v}$  on  $\mathbf{u}(\Omega)$ .

**Proof:** By the invariance of domain  $\mathbf{u}(\Omega)$  is open and  $\mathbf{u} : \Omega \rightarrow \mathbf{u}(\Omega)$  is a homeomorphism. Let  $\mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^n)$  be a vector field with  $\operatorname{div} \mathbf{v} = 0$ . For each  $y \in \mathbf{u}(\Omega)$ , consider the unique smooth flow  $\phi(y, \cdot) : \mathbb{R} \rightarrow \mathbf{u}(\Omega)$  given by

$$(3.7) \quad \frac{d\phi}{dt}(y, t) = \mathbf{v}(\phi(y, t)) \quad \text{in } \mathbb{R}, \quad \phi(y, 0) = y.$$

Using the relations  $\frac{\partial}{\partial P_j^i} \det P = (\operatorname{cof} P)_j^i$  and  $P (\operatorname{cof} P)^t = Id_n \det P$ , by a direct calculations we observe that

$$(3.8) \quad \frac{d}{dt} (\det \nabla_y \phi(y, t)) = \det \nabla_y \phi(y, t) \operatorname{div} \mathbf{v} = 0.$$

Since  $\det \nabla_y \phi(y, 0) = 1$ , from (3.8) it follows that  $\det \nabla_y \phi(y, t) = 1$  for all  $t \in \mathbb{R}$  and  $y \in \mathbf{u}(\Omega)$ . Consider the map  $\mathbf{w} : \Omega \times \mathbb{R} \rightarrow \mathbf{u}(\Omega)$  defined by

$$\mathbf{w}(x, t) := \phi(\cdot, t) \circ \mathbf{u}(x) = \phi(\mathbf{u}(x), t) \quad \text{for any } t \in \mathbb{R}, x \in \Omega.$$

Let  $V := \operatorname{supp} \mathbf{v} \subset \mathbf{u}(\Omega)$ , then  $\mathbf{v}(\mathbf{u}(x)) = 0$  for  $\mathbf{u}(x) \notin V$ . This in conjunction with the uniqueness of  $\phi$  implies that  $\phi(\mathbf{u}(x), t) = \mathbf{u}(x)$  for all points  $x$  such that  $\mathbf{u}(x) \notin V$ . Since  $\Omega$  is bounded,  $\mathbf{u}$  is continuous and  $V$  is compact,  $\Omega' = \mathbf{u}^{-1}(V)$  is a compact subset of  $\Omega$ . Hence  $\operatorname{supp}(\mathbf{w}(x, t) - \mathbf{u}(x)) \subset \Omega'$ . Furthermore,  $\det \nabla_x \mathbf{w}(x, t) = \det \nabla_y \phi(y, t) \det \nabla \mathbf{u}(x) = 1$ . Therefore,  $\mathbf{w}(\cdot, t) \in \mathcal{A}$  and  $\operatorname{supp}(\mathbf{u} - \mathbf{w}(\cdot, t)) \subset \Omega$  for all  $t \in \mathbb{R}$ . Since  $\mathbf{u}$  is a local minimizer of  $E[\cdot]$ ,

$$E[\mathbf{u}] \leq E[\mathbf{w}(\cdot, t)] \quad \text{for all } t \in \mathbb{R}.$$

Thus in particular,

$$\begin{aligned} 0 &= \left. \frac{d}{dt} \int_{\Omega} L(\nabla \mathbf{w}(x, t)) dx \right|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_j^i(\nabla \mathbf{w}(x, t)) \frac{d}{dt} \left( \frac{\partial w^i}{\partial x_j}(x, t) \right) dx \Big|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_j^i(\nabla \mathbf{w}(x, t)) \frac{\partial}{\partial x_j} \left( \frac{d\phi^i}{dt}(\mathbf{u}(x), t) \right) dx \Big|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_j^i(\nabla \mathbf{w}(x, t)) \frac{\partial}{\partial x_j} (v^i(\phi(\mathbf{u}(x), t))) dx \Big|_{t=0} \\ &= \sum_{i,j=1}^2 \int_{\Omega} L_j^i(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_j} (v^i(\mathbf{u}(x))) dx \\ &= \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx, \end{aligned}$$

for all smooth, compactly supported and divergence free vector fields on  $\mathbf{u}(\Omega)$ , where  $L_j^i(P) := \frac{\partial L}{\partial p_j^i}(P)$ . This proves the Theorem.  $\square$

**Proof of Theorem 3.1:** Let  $1 \leq r < \infty$  and  $U' \subset\subset U$ . Let  $\mathbf{u} \in \mathcal{A}$  be a local minimizer of  $E[\cdot]$  such that  $|\nabla \mathbf{u}|^2 \in h^r$  and  $|\text{cof } \nabla \mathbf{u}|^2 \in h^r(U')$  for some  $1 \leq r < \infty$ . Assume further that  $\mathbf{u} : \Omega \rightarrow \mathbf{u}(\Omega)$  is continuous and bijective map.

Now define  $\mathbf{g} = (g^1, \dots, g^n) : C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n) \rightarrow \mathbb{R}$  by

$$(3.9) \quad \langle \mathbf{g}, \mathbf{v} \rangle := \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx,$$

for all  $\mathbf{v} = (v^1, \dots, v^n) \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n)$ . In view of the volume constraint and growth condition (3.3), it follows that

$$(3.10) \quad |\langle \mathbf{g}, \mathbf{v} \rangle| \leq C(1 + \|\nabla \mathbf{u}\|_{L^2(\Omega)} + \|\text{cof } \nabla \mathbf{u}\|_{L^2(\Omega)}) \|\nabla \mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))},$$

for any  $\mathbf{v} \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n)$ . Hence  $\mathbf{g}$  is a continuous linear functional on  $C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n)$ . Using the first variation (3.6), we conclude that

$$(3.11) \quad \langle \mathbf{g}, \mathbf{v} \rangle = 0 \quad \forall \mathbf{v} \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n), \text{ div } \mathbf{v} = 0.$$

Hence there exists  $q \in \mathcal{D}'(\mathbf{u}(\Omega))$  ( see [Te 01, Proposition 1.1, p10]), such that

$$(3.12) \quad \mathbf{g} = -\nabla q \quad \text{in } \mathcal{D}'(\mathbf{u}(\Omega), \mathbb{R}^n)$$

modulo translation of a constant. In order to obtain  $h^r$  estimates of  $q$ , for  $1 \leq i, j \leq n$ , let us define  $\sigma_j^i : \Omega \rightarrow \mathbb{R}$  by

$$(3.13) \quad \sigma_j^i(x) := \sum_{k=1}^n L_k^i(\nabla \mathbf{u}(x)) \frac{\partial u^j}{\partial x_k}(x) \quad \text{for } x \in \Omega,$$

so that, the Cauchy-Green strain tensor on  $\Omega$  is given by

$$(3.14) \quad \sigma := (\sigma_j^i) = (DL(\nabla \mathbf{u}))^t \nabla \mathbf{u}$$

Define the  $ij$ -th component of the Cauchy-Green Strain tensor  $\tilde{\sigma}_j^i$  on the deformed domain  $\mathbf{u}(\Omega)$  by

$$(3.15) \quad \tilde{\sigma}_j^i := \sigma_j^i \circ \mathbf{u}^{-1} \quad \text{on } \mathbf{u}(\Omega), \quad i, j = 1, \dots, n.$$

The growth condition  $|\sigma_j^i| \leq C(|\nabla \mathbf{u}|^2 + |\text{cof } \nabla \mathbf{u}|^2)$  and  $|\nabla \mathbf{u}|^2, |\text{cof } \nabla \mathbf{u}|^2 \in L \log L$  yields  $\tilde{\sigma}_j^i \in h^1(V)$ . If  $\mathbf{u} \in K_{\text{loc}}^{1,2r}(\Omega, \mathbb{R}^n)$ ,  $1 < r < \infty$ , from the definition of  $\sigma_j^i$ ,  $\tilde{\sigma}_j^i$ , and the condition (3.3) on  $L$ , it follows that

$$(3.16) \quad \begin{aligned} \int_V |(\tilde{\sigma}_j^i)|^r &= \int_{\mathbf{u}^{-1}(V)} |\sigma_j^i|^r \\ &\leq C \left( \|\nabla \mathbf{u}\|_{L^{2r}(\mathbf{u}^{-1}(V))}^{2r} + \|\text{cof } \nabla \mathbf{u}\|_{L^{2r}(\mathbf{u}^{-1}(V))}^{2r} \right), \end{aligned}$$

for any  $V \subset\subset \mathbf{u}(\Omega)$ . Therefore, if  $|\nabla \mathbf{u}|^2 \in h^r$  and  $|\operatorname{cof} \nabla \mathbf{u}|^2 \in h_{\text{loc}}^r$  for some  $1 \leq r < \infty$ , from (3.16), we have

$$\sigma := (\sigma_j^i) \in h_{\text{loc}}^r(\Omega, \mathbb{M}^{n \times n}) \quad \text{and} \quad \tilde{\sigma} := (\tilde{\sigma}_j^i) \in h_{\text{loc}}^r(\mathbf{u}(\Omega), \mathbb{M}^{n \times n}).$$

Observe that, from the definition of  $\mathbf{g}$  in (3.9),  $\sigma_j^i$  in (3.13),  $\tilde{\sigma}_j^i$  in (3.15), and change of variables,

$$\begin{aligned} (3.17) \quad \langle \mathbf{g}, \mathbf{v} \rangle &= \sum_{i,k=1}^n \int_{\Omega} L_k^i(\nabla \mathbf{u}(x)) \frac{\partial}{\partial x_k} (v^i \circ \mathbf{u})(x) dx \\ &= \sum_{i,j,k=1}^n \int_{\Omega} L_k^i(\nabla \mathbf{u}(x)) \frac{\partial v^i}{\partial y_j}(\mathbf{u}(x)) \frac{\partial u^j}{\partial x_k}(x) dx \\ &= \sum_{i,j=1}^n \int_{\Omega} \sigma_j^i(x) \frac{\partial v^i}{\partial y_j}(\mathbf{u}(x)) dx \\ &= \int_{\Omega} \sigma(x) : \nabla_{\mathbf{u}} \mathbf{v}(\mathbf{u}(x)) dx \\ &= \int_{\mathbf{u}(\Omega)} \tilde{\sigma}(y) : \nabla \mathbf{v}(y) dy \\ &= -\langle \operatorname{div} \tilde{\sigma}, \mathbf{v} \rangle \end{aligned}$$

for any  $v \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n)$ . Hence

$$(3.18) \quad \mathbf{g} = -\operatorname{div} \tilde{\sigma} \quad \text{in } \mathcal{D}'(\mathbf{u}(\Omega), \mathbb{M}^{n \times n})$$

where the divergence is taken in each rows. Therefore, combining (3.12) and (3.18), we get

$$(3.19) \quad \nabla q = \operatorname{div} \tilde{\sigma} \quad \text{in } \mathcal{D}'(\mathbf{u}(\Omega), \mathbb{M}^{n \times n}).$$

By taking  $\mathbf{f} = \tilde{\sigma}$ , and  $U = V \subset\subset \mathbf{u}(\Omega)$  in (3.19), from Theorem 2.2, we conclude that  $q \in h_{\text{loc}}^r(\mathbf{u}(\Omega))$ , it satisfies the local representation (3.5), and

$$(3.20) \quad \begin{aligned} \|q\|_{h^r(V)} &\leq C \|\tilde{\sigma}\|_{h^r(V)} \\ &\leq C \left( \|\nabla \mathbf{u}\|^2_{h^r(\mathbf{u}^{-1}(V))} + \|\operatorname{cof} \nabla \mathbf{u}\|^2_{h^r(\mathbf{u}^{-1}(V))} \right), \end{aligned}$$

for any  $V \subset\subset \mathbf{u}(\Omega)$ , for some  $C > 0$ , depending on  $r, V, n$  and  $\mathbf{u}(\Omega)$ . Since  $q \in L_{\text{loc}}^1$ , from (3.12), it follows that

$$\langle \mathbf{g}, \mathbf{v} \rangle = -\langle \nabla q, \mathbf{v} \rangle = \langle q, \operatorname{div} \mathbf{v} \rangle = \int_{\mathbf{u}(\Omega)} q(y) \operatorname{div} \mathbf{v}(y) dy.$$



for any  $\mathbf{v} \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n)$ . Hence

$$(3.21) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v} \circ \mathbf{u})(x) dx = \int_{\mathbf{u}(\Omega)} q(y) \operatorname{div} \mathbf{v}(y) dy,$$

for any  $\mathbf{v} \in C_0^1(\mathbf{u}(\Omega), \mathbb{R}^n)$ . This proves the Theorem.  $\square$

#### 4. DERIVATION OF EULER-LAGRANGE EQUATIONS

**Theorem 4.1.** *Let  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 2$ , be a smooth, simply connected and bounded domain. Let  $\mathbf{u} \in \mathcal{A} \cap K_{\text{loc}}^{1,s}(\Omega, \mathbb{R}^n)$  for some  $s \geq 3$  be a continuous and injective local minimizer of  $E[\cdot]$ . Then the hydrostatic pressure  $p := q \circ \mathbf{u} \in L_{\text{loc}}^{s/2}(\Omega)$ , and the pair  $(\mathbf{u}, p)$  satisfies*

$$(4.1) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\Omega} p(x) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx,$$

for all  $\phi \in C_0^1(\Omega, \mathbb{R}^n)$ , where  $q \in L_{\text{loc}}^{s/2}(\mathbf{u}(\Omega))$  as in Theorem 3.1. In other words, the pair  $(\mathbf{u}, p)$  satisfies the system of Euler-Lagrange equations

$$\operatorname{div} [DL(\nabla \mathbf{u}(x)) - p(x) \operatorname{cof}(\nabla \mathbf{u}(x))] = 0 \quad \text{in } \Omega,$$

in the sense of distribution, where the divergence is taken in each rows.

**Proof.** Let  $\Omega \subset \mathbb{R}^n$  be a smooth, simply connected domain. Recall that  $K^{1,s} := \{\mathbf{w} \in W^{1,s} : \operatorname{cof} \nabla \mathbf{w} \in L^s\}$  and  $\mathcal{A} := \{\mathbf{w} \in K^{1,2}(\Omega, \mathbb{R}^n) : \det \nabla \mathbf{w} = 1 \text{ a.e.}\}$ . Let  $\mathbf{u} \in \mathcal{A} \cap K_{\text{loc}}^{1,s}(\Omega, \mathbb{R}^n)$ ,  $s \geq 3$  be a continuous injective local minimizer of the functional  $E[\cdot]$ . By Theorem 3.1, there exists  $q \in L_{\text{loc}}^{s/2}$  such that the pair  $(\mathbf{u}, q)$  satisfies the identity (3.21). Let  $\mathbf{u}^{-1} : \mathbf{u}(\Omega) \rightarrow \Omega$  be the inverse of  $\mathbf{u}$ . Then using the volume-constraint we obtain

$$\nabla_y \mathbf{u}^{-1}(y) = (\nabla_x \mathbf{u}(x))^{-1} = (\operatorname{cof} \nabla \mathbf{u}(x))^t, \quad y = \mathbf{u}(x),$$

and hence by the change of variables

$$\int_{\mathbf{u}(\Omega)} |\nabla \mathbf{u}^{-1}(y)|^2 dy = \int_{\Omega} |\operatorname{cof} \nabla \mathbf{u}(x)|^2 dx < \infty.$$

Using the relation  $\operatorname{cof}(XY) = \operatorname{cof} X \operatorname{cof} Y$ , for  $X, Y \in \mathbb{M}^{n \times n}$ , observe that

$$Id_n = \operatorname{cof}(\nabla_y \mathbf{u}^{-1} \nabla \mathbf{u}) = \operatorname{cof}(\nabla_y \mathbf{u}^{-1}) \operatorname{cof}(\nabla \mathbf{u}) = \operatorname{cof}(\nabla_y \mathbf{u}^{-1}) (\nabla \mathbf{u})^{-t},$$

and hence

$$\operatorname{cof}(\nabla \mathbf{u}^{-1}) = (\nabla \mathbf{u})^t.$$

Since  $\mathbf{u} \in K_{\text{loc}}^{1,s}(\Omega, \mathbb{R}^n)$ , it follows that  $\mathbf{u}^{-1} \in K_{\text{loc}}^{1,s}(\mathbf{u}(\Omega), \Omega)$  for  $s \geq 3$ . Let  $V \subset\subset \mathbf{u}(\Omega)$  and  $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^n)$ . Then the composition  $\phi \circ \mathbf{u}^{-1} \in$

$W_0^{1,s}(V, \mathbb{R}^n)$ . Hence there exists  $\mathbf{v}_\varepsilon \in C_0^1(V, \mathbb{R}^n)$  such that  $\mathbf{v}_\varepsilon \rightarrow \psi := \phi \circ \mathbf{u}^{-1}$  strongly in  $W^{1,s}(V, \mathbb{R}^n)$  as  $\varepsilon \rightarrow 0$ . Let  $U := \mathbf{u}^{-1}(V)$ . Then Hölder inequality yields

$$\begin{aligned} \int_U DL(\nabla \mathbf{u}) : \left( \nabla(\mathbf{v}_\varepsilon \circ \mathbf{u}) - \nabla(\psi \circ \mathbf{u}) \right) dx \\ = \int_U (\nabla \mathbf{u})^t DL(\nabla \mathbf{u}) : \left( \nabla_z \mathbf{v}_\varepsilon(\mathbf{u}) - \nabla_z \psi(\mathbf{u}) \right) dx \\ \leq C \|\nabla \mathbf{u}\|_{L^{2s'}(U)} \|\nabla(\mathbf{v}_\varepsilon - \psi)\|_{L^s(V)}, \end{aligned}$$

where  $s' := s/(s-1)$ . Notice that  $s \geq 3$  yields  $2s' \leq s$  and hence  $\nabla \mathbf{u} \in L_{\text{loc}}^s(\Omega) \subseteq L_{\text{loc}}^{2s'}(\Omega)$ . Therefore, from (3.9) we obtain

$$\begin{aligned} (4.2) \quad \langle \mathbf{g}, \mathbf{v}_\varepsilon \rangle &= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}_\varepsilon \circ \mathbf{u})(x) dx \\ &\rightarrow \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\phi \circ \mathbf{u}^{-1} \circ \mathbf{u})(x) dx \quad \text{as } \varepsilon \rightarrow 0 \\ &= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx. \end{aligned}$$

Since  $\nabla \mathbf{u}, \text{cof } \nabla \mathbf{u} \in L_{\text{loc}}^s$ ,  $q \in L_{\text{loc}}^{s/2}$  and  $L_{\text{loc}}^{s/2} \subseteq L_{\text{loc}}^{s/(s-1)}$  for  $s \geq 3$ , applying change of variables in (3.21), and letting  $\varepsilon \rightarrow 0$  we obtain

$$\begin{aligned} (4.3) \quad \langle \mathbf{g}, \mathbf{v}_\varepsilon \rangle &= \int_V q(y) \text{trace}(\nabla \mathbf{v}_\varepsilon(y)) dy \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{trace}\left(\nabla_{\mathbf{u}} \mathbf{v}_\varepsilon(\mathbf{u}(x))\right) dy \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{trace}\left(\nabla(\mathbf{v}_\varepsilon \circ \mathbf{u})(x) (\text{cof } \nabla \mathbf{u}(x))^t\right) dx \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{cof}(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}_\varepsilon \circ \mathbf{u})(x) dx, \\ &\rightarrow \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{cof}(\nabla \mathbf{u}(x)) : \nabla(\phi \circ \mathbf{u}^{-1} \circ \mathbf{u})(x) dx \\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx. \end{aligned}$$

Hence from (4.2) and (4.3) we obtain

$$\int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x)) \text{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx,$$

for any  $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^n)$ . Finally choose a sequence of smooth, simply connected sets  $V_k \subset \subset V_{k+1} \subset \subset \mathbf{u}(\Omega)$  sub-domains such that  $\mathbf{u}(\Omega) = \bigcup_{k=1}^{\infty} V_k$ . Utilizing the foregoing arguments, there exists  $q_k \in L^{s/2}(V_k)$ ,  $k \geq 1$  such that

$$(4.4) \quad \int_{\mathbf{u}^{-1}(V_k)} DL(\nabla \mathbf{u}) : \nabla \phi = \int_{\mathbf{u}^{-1}(V_k)} q_k(\mathbf{u}) \operatorname{cof}(\nabla \mathbf{u}) : \nabla \phi,$$

for  $\phi \in C_0^1(\mathbf{u}^{-1}(V_k), \mathbb{R}^n)$ . Since  $\mathbf{u}$  is locally volume-preserving homeomorphism,  $\Omega = \bigcup_{k=1}^{\infty} \mathbf{u}^{-1}(V_k)$  is an open covering of  $\Omega$  and  $\mathbf{u}^{-1}(V_k) \subset \subset \mathbf{u}^{-1}(V_{k+1})$ . Using the identity  $\operatorname{div} \operatorname{cof} \nabla \mathbf{u}(x) = 0$  and invertibility of  $\nabla \mathbf{u}(x)$ , from (4.4) it follows that  $q_k$  is unique up to a translation of a constant. Thus adding constant terms as necessary to each  $q_k$ , we deduce from (4.4) that for each fixed  $k \geq 1$

$$q_i(z) = q_k(z) \quad \text{for } z \in V_i, \quad 1 \leq i \leq k.$$

We finally define  $q : \mathbf{u}(\Omega) \rightarrow \mathbb{R}$  as  $q(z) := q_k(z)$ , for  $z \in V_k$ , so that  $q \in L_{\text{loc}}^{s/2}(\mathbf{u}(\Omega))$ . This proves that for any  $\phi \in C_0^1(\Omega, \mathbb{R}^n)$ , the pair  $(\mathbf{u}, q)$  satisfies

$$\int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\Omega} q(\mathbf{u}(x)) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx.$$

Now let us define the pressure  $p$  on  $\Omega$  by

$$p(x) := q(\mathbf{u}(x)) \quad \text{for } x \in \Omega.$$

Then for any  $k \geq 1$ ,

$$\int_{\mathbf{u}^{-1}(V_k)} |p(x)|^{s/2} = \int_{\mathbf{u}^{-1}(V_k)} |q(\mathbf{u}(x))|^{s/2} dx = \int_{V_k} |q(z)|^{s/2} dz < \infty,$$

and hence  $p \in L_{\text{loc}}^{s/2}(\Omega)$  and the pair  $(\mathbf{u}, p)$  satisfies

$$(4.5) \quad \int_{\Omega} DL(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx = \int_{\Omega} p(x) \operatorname{cof}(\nabla \mathbf{u}(x)) : \nabla \phi(x) dx,$$

for any  $\phi \in C_0^1(\Omega, \mathbb{R}^n)$ . In other words,  $(\mathbf{u}, p)$  satisfies the system of Euler-Lagrange equations

$$\operatorname{div} [DL(\nabla \mathbf{u}(x)) - p(x) \operatorname{cof}(\nabla \mathbf{u}(x))] = 0, \quad \text{in } \Omega.$$

in the sense of (4.5). This completes the proof.  $\square$

## 5. PARTIAL REGULARITY OF AREA-PRESERVING MINIMIZERS

For  $n = 2$ , as a consequence of the Euler-Lagrange equations (1.7), together with the standard elliptic estimates [GM 79], we establish the following theorem.

**Theorem 5.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a smooth, bounded simply connected domain and let  $L : \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$  be smooth, uniformly convex, such that  $DL$  has linear growth and  $D^2L$  is bounded. Let  $\mathbf{u} \in W^{1,3}(\Omega, \mathbb{R}^2)$  be an area-preserving minimizer of the energy  $E[\cdot]$ . Furthermore, assume that the associated hydrostatic pressure  $q$  on the deformed domain  $\mathbf{u}(\Omega)$  is  $C^{0,\alpha}$  for some  $0 < \alpha < 1$ . Then  $\nabla \mathbf{u}$  are Hölder continuous on a dense open set  $\Omega_0 \subset \Omega$ .*

**Proof.** Since  $\mathbf{u} \in W^{1,3}(\Omega, \mathbb{R}^2)$  and  $\mathbf{u}$  is area-preserving,  $\mathbf{u}(\Omega)$  is open and  $\mathbf{u}$  is a homeomorphism from  $\Omega$  to  $\mathbf{u}(\Omega)$ . By Theorem 4.1, there exists  $q \in L_{\text{loc}}^{3/2}(\mathbf{u}(\Omega))$  and the pair  $(\mathbf{u}, q \circ \mathbf{u})$  satisfies the system

$$(5.1) \quad \sum_{j=1}^2 \frac{\partial}{\partial x_j} \left( \frac{\partial L}{\partial p_j^i} (\nabla \mathbf{u}) - p(x) (\text{cof } \nabla \mathbf{u})_j^i \right) = 0, \quad \text{in } \Omega, \quad i = 1, 2,$$

where  $p := q \circ \mathbf{u}$ . Assume that  $q \in C^{0,\alpha}(\mathbf{u}(\Omega))$ . Since  $\mathbf{u} \in W^{1,3}$ , Sobolev imbedding theorem yields  $\mathbf{u} \in C^{1/3}$ , and hence  $p(x) = q(\mathbf{u}(x))$  is Hölder continuous with the exponent  $\alpha/3$ . Let  $F : \Omega \times \mathbb{M}^{2 \times 2} \rightarrow \mathbb{R}$  be the free-energy defined as

$$F(x, P) := L(P) - p(x) \det P \quad x \in \Omega, \quad P \in \mathbb{M}^{2 \times 2},$$

so that we can rewrite the nonlinear system (5.1) as

$$(5.2) \quad \sum_{j=1}^2 \frac{\partial}{\partial x_j} (A_j^i(x, \nabla \mathbf{u})) = 0, \quad \text{in } \Omega, \quad i = 1, 2,$$

where

$$A_j^i(x, P) := \frac{\partial F}{\partial p_j^i}(x, P) = \frac{\partial L}{\partial p_j^i}(P) - p(x) (\text{cof } P)_j^i.$$

Let  $U \subset\subset \Omega$ . Since  $|\text{cof } P| = |P|$  for any  $P \in \mathbb{M}^{2 \times 2}$ ,  $|DL(P)| \leq C(1 + |P|)$  and  $D^2L(P)$  is bounded,

$$(5.3) \quad |A_j^i(x, P)| \leq C(1 + |P|), \quad \left| \frac{\partial A_j^i}{\partial p_l^k}(x, P) \right| \leq C,$$

for any  $x \in U$ ,  $P \in \mathbb{M}^{2 \times 2}$ . By Hölder continuity of  $p$ , it follows that

$$(5.4) \quad \frac{|A_j^i(x, P) - A_j^i(y, P)|}{1 + |P|} = |p(x) - p(y)| \frac{|(\operatorname{cof} P)_j^i|}{1 + |P|} \leq C|x - y|^{\alpha/3},$$

for any  $x \in U$ ,  $P \in \mathbb{M}^{2 \times 2}$ . By direct calculations and the ellipticity of  $L$  it follows that

$$(5.5) \quad \begin{aligned} \frac{\partial A_j^i}{\partial p_l^k}(x, P) \xi_{ij} \xi_{kl} &= \frac{\partial^2 F}{\partial p_j^i \partial p_l^k}(x, P) \xi_{ij} \xi_{kl} \\ &= \frac{\partial^2 L}{\partial p_j^i \partial p_l^k}(P) \xi_{ij} \xi_{kl} - 2p(x) \det \xi \\ &\geq \lambda_0 |\xi|^2 - 2p(x) \det \xi \\ &:= I(x, \xi), \quad \text{for } P = (p_j^i), \xi = (\xi_{ij}) \in \mathbb{M}^{2 \times 2}, \end{aligned}$$

where  $\lambda_0 > 0$  is the ellipticity constant of  $L$ . Completing squares, observe that

$$(5.6) \quad \begin{aligned} \frac{I(x, \xi)}{\lambda_0} &= |\xi|^2 - 2 \frac{p(x)}{\lambda_0} \det \xi \\ &= \xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2 - 2 \frac{p}{\lambda_0} (\xi_{11} \xi_{22} - \xi_{12} \xi_{21}) \\ &= \left( \xi_{11} - \frac{p}{\lambda_0} \xi_{22} \right)^2 + \left( \xi_{12} - \frac{p}{\lambda_0} \xi_{21} \right)^2 \\ &\quad + \left( 1 - \frac{p^2}{\lambda_0^2} \right) (\xi_{22}^2 + \xi_{21}^2). \end{aligned}$$

Similarly, we obtain

$$(5.7) \quad \frac{I(x, \xi)}{\lambda_0} = \left( \xi_{22} - \frac{p}{\lambda_0} \xi_{11} \right)^2 + \left( \xi_{21} - \frac{p}{\lambda_0} \xi_{12} \right)^2 + \left( 1 - \frac{p^2}{\lambda_0^2} \right) (\xi_{11}^2 + \xi_{12}^2)$$

Adding the identities (5.6) and (5.7), we obtain

$$(5.8) \quad \begin{aligned} 2 \frac{I}{\lambda_0} &= \left( \xi_{11} - \frac{p}{\lambda_0} \xi_{22} \right)^2 + \left( \xi_{12} - \frac{p}{\lambda_0} \xi_{21} \right)^2 \\ &\quad + \left( \xi_{22} - \frac{p}{\lambda_0} \xi_{11} \right)^2 + \left( \xi_{21} - \frac{p}{\lambda_0} \xi_{12} \right)^2 + \left( 1 - \frac{p^2}{\lambda_0^2} \right) |\xi|^2 \\ &\geq \left( 1 - \frac{p^2}{\lambda_0^2} \right) |\xi|^2. \end{aligned}$$

Thus from (5.6) and (5.8), it follows that the map  $P \mapsto A(\cdot, P)$  is *strongly elliptic* if there exists  $\mu_0 > 0$  such that

$$\frac{\partial L_j^i}{\partial p_l^k}(x, P) \xi_{ij} \xi_{kl} \geq \frac{\lambda_0}{2} \left(1 - \frac{p^2}{\lambda_0^2}\right) |\xi|^2 \geq \mu_0 |\xi|^2, \quad \text{for } x \in \Omega, P, \xi \in \mathbb{M}^{2 \times 2},$$

which is equivalent to assume that

$$(5.9) \quad p^2 \leq \lambda_0^2 - 2\lambda_0\mu_0 \implies (p - \mu_0)^2 \leq (\lambda_0 - \mu_0)^2.$$

Since  $p$  is defined up to addition of arbitrary constant, thus the inequality (5.9) is satisfied in subdomain  $U \subset\subset \Omega$  if and only if

$$(5.10) \quad \text{osc}_U p < \lambda_0.$$

Since  $p$  is Hölder continuous, the estimate (5.10) holds for any subdomain  $U \subset \Omega$  with sufficiently small diameter. Hence  $A(x, P)$  is strongly elliptic in  $P$  for each  $x \in U \subset\subset \Omega$ , for sufficiently small diameter. This proves that  $A_j^i(x, P)$  satisfies all the conditions of Giaquinta-Modica in [GM 79] on  $U \subset\subset \Omega$ , with diameter of  $U$  being small. Hence by [GM 79, Theorem 1], we conclude that  $\nabla \mathbf{u}$  is Hölder continuous on a dense open subset  $U_0$  of  $U$ . By standard covering arguments we conclude the proof.  $\square$

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## REFERENCES

- [Ba 77] Ball, J.: Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rat. Mech. Anal.* **64**, 337–403 (1977).
- [BOP 92] Bauman, P., Owen, N.C., Phillips, D.: Maximum principles and an a priori estimates for an incompressible material in nonlinear elasticity. *Comm. Partial Differential Equations* **17**, 1185–1212 (1992).
- [CZ 52] Calderón, A.P., Zygmund, A.: On the existence of certain singular integrals. *Acta Math.* **88**, 85–139 (1952).
- [CKS 93] Chang, D.C; Krantz, S.G.; Stein, E.M.:  $H^p$  theory on a smooth domain in  $\mathbb{R}^N$  and elliptic boundary value problems. *J. Funct. Anal.* **114** 286–347 (1993).
- [DL 88] R. Dautry, J. L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Volume **2** Functional and Variational Methods, Springer (1988).
- [Ev 98] Evans, L.C.: *Partial Differential Equations*. Graduate Studies in Mathematics, **19**, American Mathematical Society, 1998.
- [EG 99] Evans, L.C., Gariepy, R.F.: On the partial regularity of energy-minimizing, area-preserving maps, *Calc. Var. Partial Differential Equations* **9**, 357–372 (1999).

- [Fe 71] Fefferman, C.: Characterizations of bounded mean oscillation. *Bull. Amer. Math. Soc.* **77**, 587-588 (1971).
- [FS 72] Fefferman, C.; Stein, E.M.:  $H^p$  spaces of several variables. *Acta Mathematica*, **129**, 137-193 (1972)
- [GM 79] Giaguinta, M., Modica, G.: Almost-everywhere regularity results for solutions of nonlinear elliptic systems. *Manuscripta Math.* **28**, 109-158 (1979).
- [Go 79] Goldberg, D.: A local version of real Hardy spaces. *Duke Math. J.* **46**, 27-42 (1979).
- [GT 97] Gilbarg, D., Trudinger, N.S.: *Elliptic partial differential equations of second order*. Reprint of the 1998 edition. Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [TO 81] LeTallec, P., Oden, J.T.: Existence and characterization of hydrostatic pressure in finite deformations of incompressible elastic bodies. *J. Elasticity* **11**, 341-357 (1981).
- [Mi 90]  $H^p$  spaces over open subsets of  $\mathbb{R}^n$ . *Studia Math.* **95**, 205-228 (1990).
- [Mo 52] Morrey, C.B.: Quasiconvexity and the semicontinuity of multiple integrals. *Pacific. J. Math.* **2**, 25-52 (1952).
- [Og 84] Ogden, R.W.: *Non-linear elastic deformations*. Ellis Horwood Ltd. Chichester, (1984).
- [St 70] Stein, E.: *Singular integrals and differentiability properties of functions*. Princeton University Press, Princeton, NJ, (1970).
- [St 93] Stein, E. M. *Harmonic analysis: Real-variable methods, Orthogonality, and Oscillatory integrals*. Princeton Mathematical Series; 43, Princeton University Press, Princeton, NJ, (1993).
- [Sv 88] Šverák, V.: Regularity properties of deformations with finite energy. *Arch. Rat. Mech. Anal.* **100**, 105-127 (1988).
- [Te 01] R. Temam: *Navier-Stokes Equations: Theory and Numerical Analysis*. AMS Chelsea Publishing, AMS, Rhode Island, (2001).

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